Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

E. H. Moore, of the University of Chicago, C. N. Moore, of the University of Cincinnati, D. E. Smith, of Columbia University, and H. W. Tyler, of the Massachusetts Institute of Technology, representing the colleges; and Miss Vevia Blair, of the Horace Mann School, New York, Mr. G. W. Evans, of the Charlestown High School, Boston, and Mr. Raleigh Schorling, of the Lincoln School, New York, representing the secondary schools. Two or three additional members representing secondary schools will be added to the Committee in the near future.

Two specific problems face the Committee: (1) the revision of secondary school and college courses in mathematics; (2) the revision of college entrance requirements in mathematics. The latter problem has been referred to the Committee not only by the Mathematical Association of America but also by the Council of the American Mathematical Society.

Detailed plans of the Committee are in the formative stage. Certain of its functions seem to be clear now, however. It must complete the reports already under way, it must formulate for discussion general principles which are to govern the proposed revisions (1) and (2) referred to above, it must elaborate, also for discussion, the detailed application of such general principles, it must establish intimate contact with all organizations and agencies throughout the country having similar problems in hand, it must do all it can to organize a nation-wide discussion of these problems and seek to coordinate the results of such discussion. Pending the adoption of definite plans for action the chairman or vice-chairman will welcome any suggestions looking toward the most effective methods of procedure.

A THEORY AND GENERALIZATION OF THE CIRCULAR AND HYPERBOLIC FUNCTIONS.

By A. F. FRUMVELLER, Marquette University.

There is a striking lack of symmetry and elegance in the usual treatment of hyperbolic functions; the formulas are one-sided, some admitting i, others not, in a manner apparently arbitrary. An air of unreality is added by the fact that they are made to depend essentially upon

$$e^{ix} = \cos x + i \sin x = \sum_{\infty} \frac{(ix)^n}{n!},$$

whose small geometric content entirely evaporates after a few transformations; many authorities frankly abandon the geometric phase altogether, and treat the whole topic as an exercise in the algebra of complex numbers. Long ago the writer became convinced that the one-sidedness of the formulas with respect to i arose from building the theory on too narrow a foundation; the circle after all is not a primary curve, but merely an ellipse in which the axes have accidentally become equal. If this be so, circle trigonometry is only a minor case of the trigonometry of the ellipse; and a true insight into the relation between the circular and hyperbolic functions can be attained only by moving the ellipse into the place of honor, and studying the functions that naturally present themselves in connection with its equation.

The results of an investigation into this matter proved to be highly interesting. First of all, the primary or ancestral forms for both circular and hyperbolic functions were brought to light; and secondly, the trigonometry of all the central conics, real or imaginary, presented itself in a systematic manner, and without reference to De Moivre's theorem. In fact, the ancestral type of De Moivre's formula was discovered at once.

A previous attempt at a generalization in this field was made by Prof. Irving Stringham in the *American Journal of Mathematics*, vol. 14, 1892, and in a little book entitled *Uniplanar Algebra* (Univ. of California Press). He lays down two entirely arbitrary definitions:

$$\sin_k w = \frac{k}{2}(e^{w/k} - e^{-w/k}), \qquad \cos_k w = \tfrac{1}{2}(e^{w/k} + e^{-w/k}).$$

These, with four others derived from them ($\tan_k$, $\cot_k$, $\sec_k$, $\csc_k$), he calls "modocyclic" functions; for $k = 1$, we get the hyperbolic functions, for $k = i$ the circular. Stringham's $\sin_k$ and $\cos_k$ are mutually unsymmetric, whereas the circle and hyperbola are symmetric in two ways; his formulas are therefore not the ultimate ancestral types; they are realizable in the curve $x^2 + y^2/k^2 = 1$. though he did not advert to this fact. Later on, Becker and Van Orstrand (*Smithsonian Tables of Hyperbolic Functions*, 1909) used this k-ellipse in developing their theory; but the presence of only one parameter makes it impossible to derive formulas in this way for the general central conic. Our formulas will have to have two parameters corresponding to the two parameters of the locus

1. *The "Protocyclic" functions.*

Consider the hyperbola

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1;$$

the area APM is given by

$$\frac{b}{a}\int_x^a \sqrt{x^2 - a^2}dx = \frac{b}{2a}\left(x\sqrt{x^2 - a^2} - a^2 \log \frac{x + \sqrt{x^2 - a^2}}{a}\right)$$

$$= OPM - OAP;$$

hence if OAP is called H, we have

$$\frac{2H}{ab} = \log\left(\frac{x}{a} + \frac{y}{b}\right) \qquad \text{or} \qquad \frac{x}{a} + \frac{y}{b} = e^{2H/ab}.$$

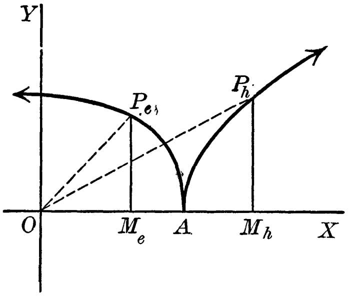
The curve-equation furnishes as the factorial mate to this

$$\frac{x}{a} - \frac{y}{b} = e^{-(2H/ab)};$$

so that

$$x_h = \frac{a}{2}\left(e^{2H/ab} + e^{-(2H/ab)}\right), \qquad y_h = \frac{b}{2}\left(e^{2H/ab} - e^{-(2H/ab)}\right).$$

Consider in the next place, the ellipse



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1;$$

its area APM is

$$\frac{b}{a}\int_a^x \sqrt{a^2 - x^2}\,dx = \frac{b}{ai}\int_a^x \sqrt{x^2 - a^2}\,dx.$$

Hence if OAP is called E, we get as above

$$\frac{2E}{ab}i = \log\left(\frac{x}{a} + \frac{iy}{b}\right),$$

and

$$x_e = \frac{a}{2}\left(e^{(2E/ab)i} + e^{-(2E/ab)i}\right), \qquad y_e = \frac{b}{2i}\left(e^{(2E/ab)i} - e^{-(2E/ab)i}\right).$$

Comparing these E and H formulas, we observe that if we put

$$2H \text{ (or } 2E) = w, \qquad \frac{1}{a} = k_1, \qquad \text{and} \qquad \frac{1}{b}\left(\text{or } \frac{i}{b}\right) = k_2,$$

the expressions for x become identical; likewise those for y; and the equation of both curves becomes $k_1^2x^2 - k_2^2y^2 = 1$. We therefore lay down the following proposition:

Every central conic $q_1^2x^2 - q_2^2y^2 = 1$ has associated with it two functions, which define x and y for any point P on the curve in terms of the area of the vertex-sector $OAP = \Omega/2$. These functions are called the cosine, and the sine, respectively; and are defined by the equations

$$(1) \qquad x = \cos\Omega = \frac{e^{q_1q_2\Omega} + e^{-q_1q_2\Omega}}{2q_1}; \qquad y = \sin\Omega = \frac{e^{q_1q_2\Omega} - e^{-q_1q_2\Omega}}{2q_2}.^{1}$$

---

[1] In the case of the circle and equilateral hyperbola this proposition has been a familiar one of our text books—for example, *An elementary treatise on the differential calculus* by J. M. Rice and W. W. Johnson, New York, 1880, pp. 139–140; according to Montucla (*Histoire des mathématiques*, Vol. 3, 1802, p. 287) the conception dates back to the time of Lambert. The extension of this idea as above to the general conic by means of an imaginary integration for the ellipse, and the consequent development of a train of two-parameter formulas including those for the circle and hyperbola as sub-cases carries this process to its ultimate completion in the most direct manner. A generalization from circle and equilateral hyperbola to any ellipse and any hyperbola was considered by C. A. Laisant, *Essai sur les fonctions hyperboliques*, Paris, 1874, p. 37 f. (also in *Mémoires de la société des sciences de Bordeaux*, Vol. 10). In particular Laisant indicates the relations between v and w (see page 286 of this paper), in which "the figure is unified in a remarkable manner." Professor Frumveller's presentation of his rediscovery of this and other relations, and of his deductions therefrom, is well worthy of perusal, even by those to whom Laisant's scarce volume is accessible. See also S. Günther, *Die Lehre von den gewöhnlichen und verallgemeinerten Hyperbelfunktionen*, 1881, p. 371 f.—EDITOR.

These, with their quotients and inverses, constitute the six functions that we shall call "protocyclic"; the number Ω will be taken to be essentially real, at least for the present, while q_1, q_2, are numbers of the general type $u + iv$. If the reference-curve be $\varphi(q_1, q_2) = 0$, the four simplest classes of these functions are those associated with the curves $\varphi(k_1, k_2)$; $\varphi(k_1, ik_2)$; $\varphi(ik_1, k_2)$; $\varphi(ik_1, ik_2)$; the two first mentioned degenerating into the ordinary hyperbolic and circular functions respectively, when $k_1 = k_2 = 1$ and both are real.

Solving equations (1) for the exponentials, and raising both sides to the nth power, we obtain the most general form of DeMoivre's theorem, valid for all n's;

$$e^{\pm q_1 q_2 n\Omega} = q_1 \cos n\Omega \pm q_2 \sin n\Omega = (q_1 \cos \Omega \pm q_2 \sin \Omega)^n. \tag{2}$$

When n is negative, we must note that $\cos \Omega = \cos(-\Omega)$, since the cosine is an even function, while the sine is odd. In like manner

$$\prod_{\lambda=1}^{n} (q_1 \cos \Omega_\lambda \pm q_2 \sin \Omega_\lambda) = q_1 \cos\left(\sum_1^n \Omega_\lambda\right) \pm q_2 \sin\left(\sum_1^n \Omega_\lambda\right), \tag{3}$$

as is evident on changing both sides into exponentials.

The equation of the curve, together with the definitions (1), gives

$$\begin{aligned} &q_1^2 \cos^2 \Omega - q_2^2 \sin^2 \Omega = 1, \text{ with its corollaries,} \\ &q_1^2 - q_2^2 \tan^2 \Omega = \sec^2 \Omega, \qquad q_1^2 \cot^2 \Omega - q_2^2 = \csc^2 \Omega. \end{aligned} \tag{4}$$

No end of formulas may now be evolved from these data! Let w and w_1 be two special values of Ω; form the four products $\sin w \cos w_1$, $\cos w \sin w_1$, $\sin w \sin w_1$, $\cos w \cos w_1$, using exponentials as in equations (1); then

$$\begin{aligned} \sin(w \pm w_1) &= q_1(\sin w \cos w_1 \pm \cos w \sin w_1), \\ \cos(w \pm w_1) &= (1/q_1)(q_1^2 \cos w \cos w_1 \pm q_2^2 \sin w \sin w_1). \end{aligned} \tag{5}$$

From these we get tangent and secant formulas, formulas for $w/2$, $2w$, $3w$, etc., in the usual way. On adding these two sine or cosine equations, and writing $\xi = w + w_1$, $\eta = w - w_1$, product-formulas result:

$$\begin{aligned} \sin \xi \pm \sin \eta &= 2q_1 \sin\left(\frac{\xi \pm \eta}{2}\right) \cos\left(\frac{\xi \mp \eta}{2}\right), \\ \cos \xi + \cos \eta &= 2q_1 \cos\left(\frac{\xi + \eta}{2}\right) \cos\left(\frac{\xi - \eta}{2}\right), \\ \cos \xi - \cos \eta &= 2\frac{q_2^2}{q_1} \sin\left(\frac{\xi + \eta}{2}\right) \sin\left(\frac{\xi - \eta}{2}\right). \end{aligned} \tag{6}$$

To find the general formulas for $\sin^m w$ and $\cos^m w$ in multiples of w, we write $\xi^r = q_1 \cos rw + q_2 \sin rw$, and $1/\xi^r = q_1 \cos rw - q_2 \sin rw$, so that

$$\cos rw = \frac{1}{2q_1}\left(\xi^r + \frac{1}{\xi^r}\right), \qquad \sin rw = \frac{1}{2q_2}\left(\xi^r - \frac{1}{\xi^r}\right).$$

Thus

$$\cos^{2n} w = \left(\frac{1}{2q_1}\right)^{2n}\left(\xi + \frac{1}{\xi}\right)^{2n}$$

$$= \left(\frac{1}{2q_1}\right)^{2n}\left[\left(\xi^{2n} + \frac{1}{\xi^{2n}}\right) + \binom{2n}{1}\left(\xi^{2n-2} + \frac{1}{\xi^{2n-2}}\right) + \cdots\right].$$

A similar expansion is found for $\sin^m w$, the results being

$$\tag{7} \begin{aligned} \cos^{2n} w &= \left(\frac{1}{2q_1}\right)^{2n-1}\left[\cos 2nw + \binom{2n}{1}\cos(2n-2)w \right.\\ &\qquad \left. + \binom{2n}{2}\cos(2n-4)w + \cdots + \frac{1}{2q_1}\binom{2n}{n}.\right] \\ \cos^{2n+1} w &= \left(\frac{1}{2q_1}\right)^{2n}\left[\cos(2n+1)w + \binom{2n+1}{1}\cos(2n-1)w \right.\\ &\qquad \left. + \cdots + \binom{2n+1}{n}\cos w\right] \\ \sin^{2n} w &= 2q_1\left(\frac{1}{2q_2}\right)^{2n}\left[\cos 2nw - \binom{2n}{1}\cos(2n-2)w \right.\\ &\qquad \left. + \binom{2n}{2}\cos(2n-4)w - \cdots + (-1)^n\frac{1}{2q_1}\binom{2n}{n}\right] \\ \sin^{2n+1} w &= \left(\frac{1}{2q_2}\right)^{2n}\left[\sin(2n+1)w - \binom{2n+1}{1}\sin(2n-1)w \right.\\ &\qquad \left. + \cdots + (-1)^n\binom{2n+1}{n}\sin w\right]. \end{aligned}$$

These will suffice to give an idea of what the new formulas look like; $q_1 = 1$, $q_2 = i$, reduces them all to the case of ordinary trigonometry.

Since our fundamental Ω-functions are exponential, they are *periodic:* their period is $2\pi i/q_1q_2$, so that

$$f(q_1q_2\Omega) = f[q_1q_2(\Omega \pm 2n\pi i/q_1q_2)]; \qquad f(q_1q_2\Omega/i) = f[q_1q_2(\Omega/i \pm 2n\pi/q_1q_2)].$$

To establish this, we observe that when $q_1q_2\Omega$ becomes $i\Omega$ by putting $q_1 = 1$, $q_2 = i$, the x and y of formulas (1) become the ordinary trigonometric functions $\cos_c$, $\sin_c$, associated with $x^2 + y^2 = 1$; thus

$$y_c = \sin_c \Omega = (1/2i)(e^{i\Omega} - e^{-i\Omega}) = \sin_c(\Omega \pm 2n\pi) = (1/2i)(e^{i(\Omega\pm 2n\pi)} - e^{-i(\Omega\pm 2n\pi)});$$

in general therefore, we get $\sin_c$, $\cos_c$, by placing $q_1q_2\Omega = i(\Omega \pm 2n\pi)$, *i. e.*, by placing

$$q_1q_2\left(\Omega \pm \frac{2n\pi i}{q_1q_2}\right) = i\Omega.$$

It is now an easy matter to follow the variation of our general functions sin, cos, tan, etc., through quarter-periods;

$$\begin{aligned}
&\text{if } q_1q_2\Omega = \frac{\pi}{2}\, i, && \left(\text{or } \Omega = \frac{\pi i}{2q_1q_2}\right), && \sin\Omega = \frac{i}{q_2}\sin_c\frac{\pi}{2} = \frac{i}{q_2}; \\
&\text{if } q_1q_2\Omega = \frac{2\pi}{2}\, i, && \left(\text{or } \Omega = \frac{2\pi i}{2q_1q_2}\right), && \sin\Omega = \frac{i}{q_2}\sin_c\frac{2\pi}{2} = 0; \\
&\text{if } q_1q_2\Omega = \frac{3\pi}{2}\, i, && \left(\text{or } \Omega = \frac{3\pi i}{2q_1q_2}\right), && \sin\Omega = \frac{i}{q_2}\sin_c\frac{3\pi}{2} = -\frac{i}{q_2}; \\
&\text{if } q_1q_2\Omega = \frac{4\pi}{2}\, i, && \left(\text{or } \Omega = \frac{4\pi i}{2q_1q_2}\right), && \sin\Omega = \frac{i}{q_2}\sin_c\frac{4\pi}{2} = 0.
\end{aligned} \tag{8}$$

The corresponding cycle for $\cos\Omega$ is $(0, -1/q_1, 0, 1/q_1)$; from these the tangent and secant follow at once. The tangent has a period $\pi i/q_1q_2$.

The combined use of formulas (5) and (8) now gives us the generalized forms of the trigonometric expressions for $\sin_c(90 \pm A)$, $\cos_c(180 \pm A)$, etc., *i.e.*, for *complements* and *supplements* of Ω; the following are specimens:

$$\begin{aligned}
&\sin\left(\Omega \pm \frac{\pi i}{2q_1q_2}\right) = \pm\frac{iq_1}{q_2}\cos\Omega; && \sin\left(\Omega \pm \frac{2\pi i}{2q_1q_2}\right) = -\sin\Omega, \\
&\cos\left(\Omega \pm \frac{\pi i}{2q_1q_2}\right) = \pm\frac{iq_2}{q_1}\sin\Omega; && \cos\left(\Omega \pm \frac{2\pi i}{2q_1q_2}\right) = -\cos\Omega.
\end{aligned} \tag{9}$$

We might mention in passing that if w is a value of Ω,

$$\frac{d}{dw}\sin w = q_1^2\cos w,$$

and

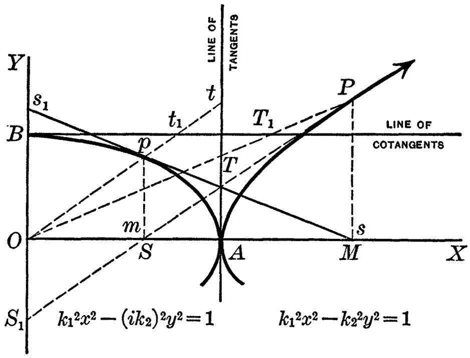
$$\frac{d}{dw}\cos w = q_2^2\sin w,$$

so that our sin and cos functions satisfy the differential equation $D_w{}^2 z - q_1{}^2 q_2{}^2 z = 0$. As a final rounding out of our theory, we give the series-expansions of these two functions:

$$\sin w = \frac{1}{q_2}\sum_0^\infty \frac{(q_1q_2w)^{2n+1}}{\underline{\left|2n+1\right.}}; \qquad \cos w = \frac{1}{q_1}\sum_0^\infty \frac{(q_1q_2w)^{2n}}{\underline{\left|2n\right.}}. \tag{10}$$

2. *Geometry of the protocyclic functions.*

If k_1, k_2, are real numbers, the reference-curve $q_1{}^2x^2 - q_2{}^2y^2 = 1$ has four fundamental types, namely $(q_1, q_2) = (k_1, k_2)$, (k_1, ik_2), (ik_1, k_2), or (ik_1, ik_2); the first and second form a pair (hyperbola and ellipse), likewise the third and fourth. Each of these types has a geometrical representation for the six functions; and the geometry of each pair has cross-relationships of a very interesting kind.



We shall use the subscripts e, h, c, for the functions connected with the ellipse, hyperbola, and circle.

(*a*) *Types* (k_1, k_2) *and* (k_1, ik_2). In the adjacent hyperbola let $w = 2(OAP)$; AT, PS, are tangents to H; $OA = 1/k_1$; $OB = 1/k_2$; $PS \equiv k_1^2xx_h - k_2^2yy_h - 1 = 0$; $OP \equiv y - (\sin_h w/\cos_h w)x = 0$. The various intercepts of these lines give at once the following table of line-values for the protocyclic functions of a point P on H:

$$\begin{array}{rlrl} PM &= \sin_h w, & OS_1 &= \dfrac{-1}{k_2^2}\csc_h w, \\[2ex] OM &= \cos_h w, & OS &= \dfrac{1}{k_1^2}\sec_h w, \\[2ex] AT &= \dfrac{1}{k_1}\tan_h w, & BT_1 &= \dfrac{1}{k_2}\cot_h w. \end{array}$$

In the same way precisely, the ellipse E has the following line-values at the point p;

$$\begin{array}{rlrl} pm &= \sin_e v, & Os_1 &= \dfrac{1}{k_2^2}\csc_e v, \\[2ex] Om &= \cos_e v, & Os &= \dfrac{1}{k_1^2}\sec_e v, \\[2ex] At &= \dfrac{1}{k_1}\tan_e v, & Bt_1 &= \dfrac{1}{k_2}\cot_e v. \end{array}$$

When p and P are chosen at random, there is no relation between v and w; but if ordinate pm be erected at the point S, the entire figure is unified in a remarkable manner.[1] We have first of all, $OS = \cos_e v = (1/k_1^2)\sec_h w$; ps, the tangent at p, has for its x-intercept, $1/(k_1^2x_e) = (1/k_1^2)\sec_e v = \cos_h w$, so that it passes through M. Using formulas (4) of § 1 on the two values of OS, we have

$$(1/k_1^2)(1 - k_2^2\sin_e^2 v) = (1/k_1^4)(k_1^2 - k_2^2\tan_h^2 w)$$

whence

$$\sin_e v = \left(\frac{1}{k_1}\right)\tan_h w;$$

applying

$$\cos_e v = \left(\frac{1}{k_1^2}\right)\sec_h w$$

to this, we get $\tan_e v = k_1\sin_h w$; which shows that the points t and P, T and p, are respectively at the same level above OX. The relation thus existing between v and w is expressed by the functional symbol $v_e = \text{am}\, w_h$, or $v_e = \text{gd}\, w_h$ (read, "v is the *amplitude* or *gudermannian*[1] of w").

---

[1] Cf. footnote on page 282.

The operation "*gd*" is carried out as follows, by means of formula 2, § 1;

$$e^{k_1k_2w} = k_1 \cosh_h w + k_2 \sin_h w = k_1^{-1} \sec_e v + (k_2/k_1) \tan_e v = V,$$

whence

$$w = \frac{1}{k_1k_2} \log V.$$

If there were in existence a table of the various functions of v_e, we could consequently compute w_h and its functions from that table; ordinary hyperbolic functions for $k_1 = k_2 = 1$ are thus easily found from the corresponding tables of circular trigonometry.[2]

(*b*) *Types* (ik_1, k_2) *and* (ik_1, ik_2). The fundamental loci now become

$$\begin{cases} H_{-1}; \quad (ik_1)^2x^2 - (ik_2)^2y^2 = 1, \quad \text{or} \quad -k_1^2x^2 + k_2^2y^2 = 1. \\ E_{-1}; \quad (ik_1)^2x^2 - \quad k_2^2y_2 \;= 1, \quad \text{or} \quad -k_1^2x^2 - k_2^2y^2 = 1. \end{cases}$$

The ellipse E_{-1} is the imaginary form of our previous E; hence H_{-1} must be regarded as the imaginary form of H. All the formulas of § 1 were predicated on the supposition that Ω was *real*; but we may generalize our point of view as regards x and y in the following manner: "*the theory as well as the geometry of the protocyclic functions remain valid, even when x or y is a pure imaginary number.*"

Graphing under such circumstances has been fully worked out by the writer in a former paper[3] to which the reader must be referred for details.

The equation H_{-1}, or $-k_1^2x^2 + k_2^2y^2 = 1$, when $x = u + iv$, and y is real, becomes $k_2^2y^2 = 1 + k^2{}_1(u^2 - v^2)$, with the condition $uv = 0$; if $v = 0$,

$$k_2^2y^2 - k_1^2u^2 = 1$$

in the plane $\overline{uoy}$; if $u = 0$, $k_2^2y^2 + k_1^2v^2 = 1$ in plane $\overline{voy}$. The other equation E_{-1}, or $k_1^2x^2 + k_2^2y^2 = -1$, gives $1 = -k_1^2u^2 - k_2^2y^2$ in $\overline{uoy}$, and $1 = k_1^2v^2 - k_2^2y^2$ in $\overline{voy}$; in the complex plane $\overline{voy}$, we therefore find the geometrical locus of case (*a*) reproduced exactly!

When the k's become complex numbers, we can still write

$$(H) \qquad y^2 = \frac{1}{k_2^2} - \left(\frac{k_1^2}{k_2^2}\right) x^2 = (r_1 + is_1) - (r_2 + is_2)(u + iv)^2$$

$$= (r_1 - r_2u^2 + r_2v^2 + 2s_2uv) + i(s_1 - s_2u^2 + s_2v^2 - 2r_2uv),$$

---

[1] The term "Gudermannian" was introduced by Cayley in a paper of 1862 (see *Collected Mathematical Papers*, Volume 5, p. 86); if $u = \log \tan [(\pi/4) + (\phi/2)]$, then $\phi = 1/i \log \tan [(\pi/4) + \frac{1}{2}ui]$ = "gd u (Gudermannian of u)." The name was given in honor of Gudermann who called the function the "longitude of u" (Crelle's *Journal*, Vol. 6, 1830, p. 165).—EDITOR.

[2] The *Smithsonian Tables* referred to above contain an extensive table of the gudermannian (and another of the antigudermannian); a brief one may be found in B. O. Peirce, *A Short Table of Integrals*, revised edition.—EDITOR.

[3] "The graph of $f(x)$ for complex numbers," AMERICAN MATH. MONTHLY, Vol. 24, November, 1917.

with a similar equation for (E), but there is no simple geometrical method of attacking this wilderness of ordinates. The y's stand at every point in the x-plane, but they have various slants towards the fourth direction of space; all we notice is, that those of the same slant or tilt have their feet on an hyperbola around O as center. Methods of function-theory must here be used.

Conclusion.—One primary advantage of treating hyperbolic functions from this general standpoint lies, in the writer's judgment, in the relegation of i to its proper subordinate position; protocyclic functions are real, and represent geometric entities from the very start; the generalization that occurs in their definition is seen to be geometrically appropriate. The theoretic interest is mainly in the fact that circle-trigonometry and hyperbola-trigonometry can now be comprehended in the wider topic of the trigonometry of the central conics.

A PROOF OF A THEOREM OF COMPOUND PROBABILITIES.

By UGO BROGGI,[1] Universidad De La Plata.

1. The definition of a geometric probability is well known. We call the probability, $\phi(a, b)$, that a real number, x, belongs to the interval (a, b) a real function, ϕ, of a and b which satisfies the conditions

$$0 \leqq \phi(a, b) \leqq 1,$$

and, if $a \leqq c \leqq b$,

$$\phi(a, b) = \phi(a, c) + \phi(c, b).$$

It would be an easy matter, but beside our purpose, to generalize the definition, and apply it not only to intervals but also to more general one- or more-dimensional sets of points.

Since the definition of our new probability is based upon the theorem of total probabilities, it is evident that it must agree with that theorem.

We cannot without proof say the same of the theorem of compound probabilities, which asserts that if p is the probability that x belongs to (a, b), and π the probability that y belongs to (α, β) and there is no relation between x and y, the probability P that the two-dimensional number (x, y) belongs to the region

$$a \leqq x \leqq b, \qquad \alpha \leqq y \leqq \beta,$$

is expressed by $p\pi$.

It is a surprising fact that nowhere do we find the demonstration of this theorem, one which is evidently fundamental and without which the theory of errors and the kinetic theory of gases would come to nothing.

We propose to give here the missing demonstration.

---

[1] Professori Broggi is a native of Italy, received his doctorate at Göttingen (1907) and has long been professor of mathematics at the University of La Plata in Buenos Aires. His dissertation was entitled *Die Axiome der Wahrscheinlichkeitsrechnung* and his work *Matematica Attuariale* (Milano, 1906) was issued in French form in 1907, and in German in 1911.—Editor.